

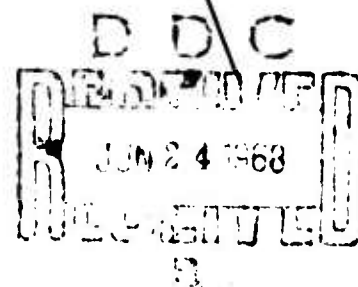
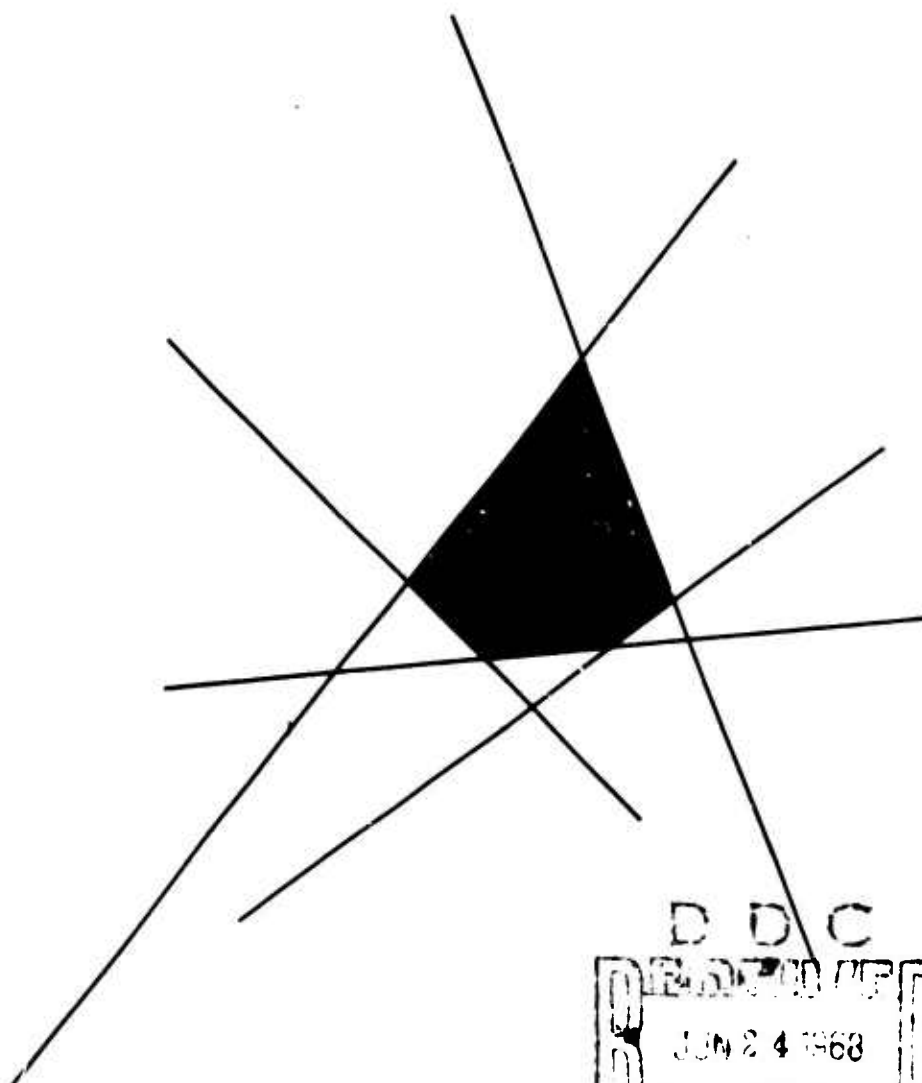
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CONTRIBUTIONS TO THE THEORY OF EXTREME VALUES

by

ROBERT HARRIS

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ABSTRACT

Extreme value distribution laws are obtained for the lifetimes of multi-component systems with replaceable components, under various assumptions on the asymptotic relationship between number of components in the system and number of spare components. Results are given for limiting distribution laws of order statistics from non-homogeneous samples and samples of random size, and applied to the superposition of renewal processes. An attempt is made to put extreme value theory into a general framework using the notion of a coherent structure, and some new results utilizing this idea are presented.

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INTRODUCTION

The limiting distribution of the maximum term in a sequence of independent, identically distributed random variables was completely analysed in a series of works by many writers, culminating in the comprehensive work of Gnedenko [6]. Results for order statistics of fixed and increasing rank were obtained by Smirnov [14], who completely characterized the limiting types and their domains of attraction. Generalizations of these results for the maximum term have been made by several writers; Juncosa [10] dropped the assumption of a common distribution, Watson [15] proved that under slight restrictions the limiting distribution of the maximum term in a stationary sequence of m -dependent random variables is the same as in the independent case, and Berman [2] studied exchangeable random variables and samples of random size. A bibliography and discussion of applications is contained in the book by Gumbel [9].

This paper extends the classical theory. The second section introduces a model from reliability theory - essentially a series system with replaceable components. It is shown that the asymptotic distribution of system lifetime can belong to one of two types when the number of spares is fixed or of a smaller order than the total number n of components, as n becomes infinite, and that these limiting distributions are the same as those obtained by Gnedenko, Chibisov [4] and Smirnov. Sections 3 and 4 deal with the limiting distribution of order statistics when the assumptions of common distribution and fixed sample size are dropped. The results of these three sections are

then applied in Section 5 to the superposition of a large number of renewal processes and compared to the necessary and sufficient conditions of Grigelionis [8] concerning closeness to a Poisson process. In particular, it is shown that randomness of the sample size leads to a point process which is a mixture of nonhomogeneous Poisson processes. Finally, in Section 6, an attempt is made to put extreme value theory into a more general framework using the notion of a coherent structure [1]. As noted there, most of the classical problems in extreme value theory are contained in the following general question: "Given a coherent structure with a finite number of components and some procedure to increase the number of components without bound, what are the possible limiting distributions for the structure lifetime for given component lifetime distributions?" For example, the minimum of a set of n random variables corresponds to the lifetime of a series structure of n components, and the limiting procedure adds one component at a time to the structure. In addition, the class of limiting distributions is characterized for the case of an arbitrary coherent structure when the procedure for expansion is that of repeated composition.

1. NOTATION AND CLASSICAL RESULTS

Throughout this paper, the distribution function of a random variable X will be denoted by $P\{X \leq x\} = F(x)$, and the tail of the distribution by $P\{X > x\} = \bar{F}(x)$. The abbreviation "d.f." will be used for distribution function. A d.f. will be called *proper* if:

$$\lim_{x \rightarrow \infty} F(x) = 1, \quad \lim_{x \rightarrow -\infty} F(x) = 0$$

and not all its mass is concentrated at one point. Two d.f.'s $F_1(x)$ and $F_2(x)$ are said to be of the *same type* if there exist constants $A > 0$ and B such that: $F_1(Ax + B) = F_2(x)$ for all values of x . Unless otherwise stated, all d.f.'s will be assumed proper and all limiting d.f.'s should be taken to mean limiting types of d.f.'s. Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables with common distribution $F(x)$, and let $\xi_n = \min(X_1, X_2, \dots, X_n)$. Then the limiting d.f. of ξ_n belongs to exactly one of three types, [6]; that is to say, if there exist sequences of normalizing constants $\{a_n > 0\}$ and $\{b_n\}$ and a d.f. $G(x)$ such that:

$$\lim_{n \rightarrow \infty} P\left\{a_n^{-1} (\xi_n - b_n) \leq x\right\} = G(x)$$

at each continuity point of $G(x)$, then $G(x)$ belongs to one of the following types:

(1.1)	$\phi_{(1)}(x) = 0$	for $x \leq 0$
	$= 1 - \exp[-x^\alpha]$	for $x > 0, \alpha > 0$
	$\phi_{(2)}(x) = 1 - \exp[-(-x)^{-\alpha}]$	for $x < 0, \alpha > 0$
	$= 1$	for $x \geq 0$
	$\phi_{(3)}(x) = 1 - \exp[-\exp x]$	$-\infty < x < \infty$

The domain of attraction of a limiting d.f. $G(x)$ is the set of all d.f.'s $F(x)$ such that for suitable choice of normalizing constants $\{a_n > 0\}$ and $\{b_n\}$

$$(1.2) \quad \lim_{n \rightarrow \infty} \bar{F}^n(a_n x + b_n) = \bar{G}(x) .$$

By a well-known theorem of Khintchine (e.g., see [7], p. 40), each d.f. can belong to at most one domain of attraction. Necessary and sufficient conditions were given by Gnedenko [6] for a d.f. to belong to the domain of attraction of $\phi_{(1)}(x)$, $\phi_{(2)}(x)$ or $\phi_{(3)}(x)$. For example, $F(x)$ is in the domain of attraction of $\phi_{(1)}(x)$ if and only if $\exists x_0$ such that $F(x_0) = 0$, $F(x_0 + \epsilon) > 0$ for each $\epsilon > 0$ and

$$(1.3) \quad \lim_{x \rightarrow 0+} \frac{F(x_0 + tx)}{F(x_0 + x)} = t^\alpha \quad \text{for all } t > 0 .$$

The k^{th} smallest variable from (X_1, X_2, \dots, X_n) will be denoted by $\xi_n^{(k)}$, so that $\xi_n^{(1)} = \xi_n$; limiting d.f.'s for these random variables as obtained by Smirnov and Chibisov will be introduced as needed.

The notation $f(x) = O(g(x))$ as $x \rightarrow a$ will mean $|f(x)/g(x)| \leq k < \infty$ as $x \rightarrow a$, and $f(x) = o(g(x))$ as $x \rightarrow a$ will be used to denote that $f(x)/g(x) \rightarrow 0$ as $x \rightarrow a$. Likewise $f(x) \sim g(x)$ as $x \rightarrow a$ implies that $f(x)/g(x) \rightarrow 1$ as $x \rightarrow a$.

$\phi(x)$ will be used for the standard normal $(0,1)$ d.f.

2. STRUCTURES WITH REPLACEMENT

The problem that is investigated here is the following: a system consists of n identical and independent components in series, with m inactive spare components available which instantaneously replace the components as they fail, until there are no more spares, whereupon the system fails. The system lifetime will be denoted by $\eta_n^{(m+1)}$, $(m+1)$ being the total number of component failures which must occur before system failure. The investigation is in two parts, corresponding to the cases when $m = m(n)$ is of a smaller order than n or of the same order as n , and a third subsection describes how some of the results may be carried over to more general types of systems. It is assumed in this section that $F(0-) = 0$.

Extreme Terms

Let $G_{nm}^*(x) = P\{\eta_n^{(m+1)} \leq x\}$. Then it is shown that the class of limiting d.f.'s for the system lifetime as $n \rightarrow \infty$, with appropriate linear norming constants, is the same as the limiting d.f.'s of the corresponding order statistics provided that m is finite or of smaller order than \sqrt{n} , as in the following two theorems.

Theorem 2.1:

The limit laws for sequences $G_{nm}^*(a_n x + b_n)$ of system lifetime d.f.'s, with m fixed, are exhausted by the following two types:

$$\begin{aligned}
 \phi_{(1)}^{(m)}(x) &= 0 && \text{for } x \leq 0 \\
 (2.1) \quad &= \frac{1}{(m-1)!} \int_0^{x^\alpha} e^{-y} y^{m-1} dy && \text{for } x > 0, \alpha > 0 \\
 \phi_{(3)}^{(m)}(x) &= \frac{1}{(m-1)!} \int_0^{e^x} e^{-y} y^{m-1} dy && -\infty < x < \infty
 \end{aligned}$$

Theorem 2.2:

If $m \sim cn^\alpha$, with $c > 0$, $0 < \alpha < \frac{1}{2}$, then the only possible limit d.f.'s for the sequence $G_{nm}^*(a_n x + b_n)$ are:

$$\begin{aligned}
 G_{(1)}(x) &= \phi(x) \\
 (2.2) \quad G_{(2)}(x) &= 0 & \text{for } x \leq 0 \\
 &= \phi(\beta \log x) & \text{for } x > 0, \beta > 0
 \end{aligned}$$

Notice that $G_{(2)}(x)$ is the log normal d.f.

Some preliminary results are needed before the proofs of Theorems 2.1 and 2.2 can be given:

$$(2.3) \quad \bar{G}_{nm}^*(x) = \sum_{j=0}^m \sum_{i_1 + \dots + i_n = j} \prod_{k=1}^n \left\{ F^{(i_k)}(x) - F^{(i_k+1)}(x) \right\}.$$

Where $F^{(k)}(x)$ is the k -fold convolution of the d.f. $F(x)$ and the inner summation is over all nonnegative combinations of (i_1, i_2, \dots, i_n) which sum to j . This formula follows from the superposition of n identical renewal processes.

The d.f. $F(x)$ will be assumed to be concentrated on the nonnegative real axis in this section since the concept of component lifetime is meaningful only in this case. Use will be made of the inequality

$$(2.4) \quad F^{(k)}(x) \leq \{F(x)\}^k, \quad \forall k \geq 1, \quad \forall x \geq 0.$$

It is convenient to speak of n "sockets" in series, each of which must contain a working component for the system to work. When m is not too large, a key step in the proofs will be to show that the probability of two or more failures in any socket is negligible as $n \rightarrow \infty$. Define

$$(2.5) \quad \bar{G}_{nm}(x) = \sum_{j=0}^m \binom{n}{j} \bar{F}^{n-j}(x) F^j(x)$$

i.e., the survival probability of an $(m+1)$ -out-of- n system.

Theorem 2.3:

If $m = o(n^{\frac{1}{2}})$ as $n \rightarrow \infty$, and if $\{a_n > 0\}$ and $\{b_n\}$ are sequences of normalizing constants such that

$$F(a_n x + b_n) = o(n^{-\frac{1}{2}}) \text{ as } n \rightarrow \infty, \quad \forall x \geq 0, \text{ then}$$

$$(2.6) \quad \lim_{n \rightarrow \infty} |\bar{G}_{nm}(a_n x + b_n) - \bar{G}_{nm}^*(a_n x + b_n)| = 0, \quad \forall x \geq 0.$$

The proof of this theorem will depend on the following lemmas.

Lemma 2.1:

- (i) The number of ways in which j failures can occur, in such a way that at most one failure occurs in each socket, is $\binom{n}{j}$.
- (ii) The total number of ways in which j failures can occur, the number of failures in any socket being arbitrary, is $\binom{n+j-1}{j}$.

Proof:

- (i) Follows by considering the coefficient of z^j in $(1+z)^n$.
- (ii) Follows by considering the coefficient of z^j in $(1+z+z^2+\dots)^n = (1-z)^{-n}$.

Lemma 2.2:

If $0 \leq j \leq m$, and $m = o(n^{\frac{1}{2}})$ as $n \rightarrow \infty$, then

$$\binom{n}{j} / \binom{n+j-1}{j} \rightarrow 1, \text{ as } n \rightarrow \infty.$$

Proof:

$$\begin{aligned}
 1 &\geq \frac{\binom{n}{j}}{\binom{n+j-1}{j}} = \prod_{k=0}^{j-1} \left(\frac{n-k}{n-k+j-1} \right) \geq \left(1 - \frac{1-1}{n} \right)^j \\
 &\geq \left(1 - \frac{m}{n} \right)^n \\
 &\geq \left(1 - \frac{\varepsilon}{\sqrt{n}} \right)^{\varepsilon\sqrt{n}}, \text{ for } n \text{ sufficiently large.} \\
 &\rightarrow \exp. [-\varepsilon^2],
 \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. The result follows on letting $\varepsilon \rightarrow 0$.

Proof of Theorem 2.3:

Define the following notation:

- 1) $A_{nj} = \binom{n+j-1}{j} - \binom{n}{j}$
- 2) $u_{nj}(x) = \bar{F}^{n-j}(x) F^j(x)$
- 3) $v_{nj}(x) = \bar{F}^{n-j}(x) \{F(x) - F^{(2)}(x)\}^j$
- 4) $w_{nj}(x)$ will be used for all terms of the form:

$$\prod_{k=1}^n \left\{ F^{(i_k)}(x) - F^{(i_k+1)}(x) \right\},$$

where $i_1 + \dots + i_n = j$ and at least one of the $i_k \geq 2$. Notice from (2.4) that:

$$(2.7) \quad 0 \leq v_{nj}(x) \leq u_{nj}(x), \quad 0 \leq w_{nj}(x) \leq u_{nj}(x).$$

Now:

$$\bar{G}_{nm}^*(x) / \bar{G}_{nm}(x) = \left\{ \sum_{j=0}^m \left[\binom{n}{j} v_{nj}(x) + A_{nj} w_{nj}(x) \right] \right\} / \sum_{j=0}^m \binom{n}{j} u_{nj}(x).$$

But:

$$\begin{aligned}
 0 &\leq \sum_{j=0}^m A_{nj} w_{nj}(x) / \sum_{j=0}^m \binom{n}{j} u_{nj}(x) \\
 &\leq \max_{j=0, \dots, m} A_{nj} w_{nj}(x) / \binom{n}{j} u_{nj}(x) \\
 &\rightarrow 0, \text{ by (2.7) and Lemma 2.2.}
 \end{aligned}$$

Also:

$$\begin{aligned}
 0 &\leq 1 - \sum_{j=0}^m \binom{n}{j} v_{nj}(x) / \sum_{j=0}^m \binom{n}{j} u_{nj}(x) \\
 &= \sum_{j=0}^m \binom{n}{j} [u_{nj}(x) - v_{nj}(x)] / \sum_{j=0}^m \binom{n}{j} u_{nj}(x) \\
 &\leq \max_{j=0, \dots, m} [u_{nj}(x) - v_{nj}(x)] / u_{nj}(x) \\
 &= 1 - [1 - F^{(2)}(x) / F(x)]^m \\
 &\leq 1 - [1 - F(x)]^m, \text{ by use of (2.4).}
 \end{aligned}$$

Now if x is replaced by $(a_n x + b_n)$ and the second assumption of the theorem used, it is seen that the last term approaches zero as $n \rightarrow \infty$.

Combining results:

$$|\bar{G}_{nm}^*(a_n x + b_n) / \bar{G}_{nm}(a_n x + b_n) - 1| \rightarrow 0, \text{ as } n \rightarrow \infty,$$

and since d.f.'s are bounded, the theorem is proved.

Proof of Theorems 2.1 and 2.2:

To examine the possible normalizing sequences $\{a_n\}$ and $\{b_n\}$ which satisfy the conditions of Theorem 2.3, it is necessary to consider separately

the cases where m remains finite or $m \rightarrow \infty$. Suppose first that m remains finite. Then Smirnov [14] has shown that in order for

$$(2.8) \quad G_{nm}(a_n x + b_n) \rightarrow G(x)$$

for suitable choice of normalizing constants, where $G(x)$ is a proper d.f., it is necessary and sufficient that

$$(2.9) \quad v_n(x) = nF(a_n + b_n) \rightarrow v(x)$$

where $v(x)$ is a nondecreasing nonnegative function defined by:

$$(2.10) \quad \frac{1}{(m-1)!} \int_0^{v(x)} e^{-y} y^{m-1} dy = G(x)$$

Furthermore, he proved that (up to a linear transformation) the function $v(x)$ must be one of the three forms x^α , $(-x)^{-\alpha}$ or e^{-x} , where α is an arbitrary positive constant. The domain of attraction corresponding to the second form for $v(x)$ consists of d.f.'s which are unbounded below, so that on using Theorem 2.3 and the nonnegativity assumption on the $\{X_i\}$, Theorem 2.1 is proved.

Now suppose that $m \sim cn^\alpha$, where $c > 0$, $0 < \alpha < \frac{1}{2}$. Chibisov [4] has shown that $G_{nm}(a_n x + b_n) \rightarrow G(x)$ if and only if

$$(2.11) \quad u_n(x) = m^{-\frac{1}{2}} \{nF(a_n x + b_n) - m\} \rightarrow u(x)$$

where $u(x)$ is defined by the equation.

$$(2.12) \quad G(x) = \phi(u(x))$$

and ϕ is the normal $(0,1)$ d.f. The function $u(x)$ must be of the same

type as one of x , $\beta \log x$ or $-\beta \log|x|$, where $\beta > 0$ is an arbitrary constant, and the domain of attraction corresponding to the third form contains only d.f.'s which are unbounded below. For a normalizing sequence which satisfies (2.11), it is clear that $F(a_n x + b_n) = O(n^{\alpha-1}) = o(n^{-\frac{1}{2}})$; thus the conditions of Theorem 2.3 are satisfied and Theorem 2.2 is proved.

Similarly, characterisations of the domains of attraction of these limit d.f.'s may be made. Note also that one might wish to restrict the limiting law itself to correspond to a nonnegative random variable, thus eliminating one of the types in Theorems 2.1 and 2.2.

The assumption that the spares have the same lifetime d.f. as the original components is unnecessary; any d.f. $F^*(x)$ such that $F^*(a_n x + b_n) = o(n^{-\frac{1}{2}})$ will suffice. The appropriate modifications to the proof of Theorem 2.3 present no difficulty.

It would be desirable to relax the restriction $\alpha < \frac{1}{2}$ which appears in the conditions of Theorem 2.2. Results may be obtained for $\alpha < 2/3$ as described by Lemmas 2.3 and 2.4, but the more general case $\alpha < 1$ does not seem amenable to analysis and a counter-intuitive reason for this is given in Lemmas 2.5 and 2.6.

Let the symbol " $\stackrel{st}{\geq}$ " stand for "stochastically greater than".

Lemma 2.3:

For independent, identically distributed nonnegative component random variables:

$$\xi_n^{(m)} \stackrel{st}{\geq} \eta_n^{(m)} \stackrel{st}{\geq} \xi_{n+m}^{(m)}$$

where the ξ 's are the corresponding order statistics.

Proof:

The first part of the inequality follows by observing the replacements themselves may fail, thus giving rise to more failures; the second part by observing that time to system failure decreases if the spares are subject to failure from the initial instant.

Lemma 2.4:

If $m \sim cn^\alpha$, with $c > 0$, $\frac{1}{2} \leq \alpha < 2/3$, then the limit d.f.'s (2.2) are possible for the sequence $G_{nm}^*(a_n x + b_n)$.

Proof:

Suppose that F and $\{a_n > 0\}$, $\{b_n\}$ are such that (2.11) and (2.12) hold, so that

$$(2.13) \quad F(a_n x + b_n) = m/n + u(x)\sqrt{m}/n + o(\sqrt{m}/n).$$

Then

$$(2.14) \quad \frac{(n+m) F(a_n x + b_n) - m}{\sqrt{m}} = u(x) + O(m^{3/2}/n) \rightarrow u(x).$$

Thus both $a_n^{-1} (\xi_n^{(m)} - b_n)$ and $a_n^{-1} (\xi_{n+m}^{(m)} - b_n)$ have the same limiting d.f. and hence by Lemma 2.3 so does $a_n^{-1} (\eta_n^{(m)} - b_n)$. Thus with Chibisov's results, the lemma is proved.

It should be noted that although Lemma 2.4 shows that the limiting d.f.'s (2.2) are possible, it does not rule out other limiting d.f.'s, in contrast to the results of Theorems 2.1 and 2.2.

Lemma 2.5:

The number of ways that m failures can occur in n sockets with at most r failures per socket is

$$c(n, m, r) = \sum_{i=0}^{\lfloor m/(r+1) \rfloor} (-1)^i \binom{n}{i} \binom{n+m-ri-i-1}{n},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

Proof:

The form of $c(n, m, r)$ follows by observing that it is the coefficient of z^m in:

$$(1 + z + z^2 + \dots + z^r)^n = (1 - z^{r+1})^n (1 - z)^{-n}.$$

Let $c(n, m) = \binom{n+m-1}{m}$ -- the total number of ways that m failures can occur in n sockets.

Lemma 2.6:

If $m \sim cn^\alpha$, where $c > 0$, $0 < \alpha < 1$, and r is fixed, then $c(n, m, r)/c(n, m) \rightarrow 1$ as $n \rightarrow \infty$ provided $r \geq \lceil 1/(1 - \alpha) \rceil$.

Proof:

Write $c(n, m, r) = a_0 - a_1 + a_2 - \dots (-1)^s a_s$, where $a_1 = \binom{n}{1} \binom{n+m-ri-i-1}{n}$ and $s = \lfloor m/(r+1) \rfloor$. Then:

$$\begin{aligned} \frac{a_1}{a_{i+1}} &= \frac{i+1}{n-i} \cdot \frac{(n+m-ri-i-1)!}{(m-ri-i-1)!} \cdot \frac{(m-ri-r-i-2)!}{(n-m-ri-r-i-2)!} \\ &\geq \frac{i+1}{n-i} \left\{ 1 + \frac{n}{m-ri-i-1} \right\}^{r+1} \\ &\geq \frac{1}{n} \left\{ 1 + \frac{n}{m-1} \right\}^{r+1} = \frac{1}{\rho(n)}, \text{ say.} \end{aligned}$$

Then $\rho(n) \rightarrow 0$ as $n \rightarrow \infty$ provided $(r+1) > (1 - \alpha)^{-1}$. Now

$$\begin{aligned}
|c(n,m,r)/c(n,m) - 1| &= |(a_0 - a_1 + a_2 - \dots (-)^s a_s)/a_0 - 1| \\
&\leq |(a_1 + a_2 + \dots + a_s)/a_0| \\
&\leq \rho + \rho^2 + \dots + \rho^s \\
&= \rho(1 - \rho^s)/(1 - \rho) \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Thus for the case $m \sim cn^\alpha$, $0 < \alpha < 1$, although it is tempting to think that one need only consider at most one failure per socket in the limit, Lemma 2.6 shows that a large number of the total ways of failure actually involve more than one failure per socket.

Central Terms

The results obtained in the first part of this section are for the limiting d.f.'s of extreme terms in which the number of spares is of a smaller order than the number of components in the system; this part treats the central terms where the numbers of spares and components are of the same order. It is shown in Theorem 2.5 that under fairly weak conditions the limiting d.f. of $a_n^{-1} (\eta_n^{(n)} - b_n)$, for appropriate choice of $a_n > 0$ and b_n , is the normal d.f. For simplicity of notation, it is assumed that $m = n - 1$ although it is obvious that Theorems 2.4 and 2.5 hold with slight modifications when $m = m(n)$ is such that $m(n)/n \rightarrow \lambda$, $0 < \lambda < \infty$.

Definition:

Following Kolmogorov [11] and Smirnov [14], a sequence $\{X_n\}$ of random variables is said to be *stable* if \exists constants a_n such that $P(|X_n - a_n| < \epsilon) \rightarrow 1$, as $n \rightarrow \infty$, for each fixed $\epsilon > 0$.

Theorem 2.4 demonstrates the stability of the sequence of system lifetime $\{\eta_n^{(n)}\}$ under mild restrictions. Some additional notation is needed;

let $N_i(t)$ denote the number of component failures in the i^{th} socket up to and including time t , $1 \leq i \leq n$, $S_n(t) = \sum_{i=1}^n N_i(t)$ the total number of failures. Set $\mu(t) = E(N_i(t))$ and $\sigma^2(t) = \text{Var}(N_i(t))$ as the mean and variance of $N_i(t)$. It is well known that renewal counting functions $N_i(t)$ have finite moments of all orders for each fixed t so that the existence of $\mu(t)$ and $\sigma(t)$ is guaranteed.

Theorem 2.4:

If $\mu(t)$ is increasing in a neighborhood of $t = \mu^{-1}(1)$, then the sequence $\{\eta_n^{(n)}\}$ is stable.

Proof:

Fix $\epsilon > 0$ and let t_1 be the unique t such that $\mu(t) = 1$. Then

$$\{t_1 - \epsilon < \eta_n^{(n)} < t_1 + \epsilon\} \Leftrightarrow \{S_n(t_1 + \epsilon)/n \geq 1 > S_n(t_1 - \epsilon)/n\}.$$

For arbitrary $\epsilon^* > 0$, $P\{|S_n(t)/n - \mu(t)| < \epsilon^*\} \rightarrow 1$, as $n \rightarrow \infty$, for all finite t , by the weak law of large numbers. Thus

$P\{S_n(t_1 + \epsilon)/n > \mu(t_1 + \epsilon) - \epsilon^*\} \rightarrow 1$, and by choosing ϵ^* sufficiently small, it is clear that $\mu(t_1 + \epsilon) - \epsilon^* \geq 1$ and so $P\{S_n(t_1 + \epsilon)/n > 1\} \rightarrow 1$.

Similarly, $P\{S_n(t_1 - \epsilon)/n < 1\} \rightarrow 1$, so that finally

$$(2.15) \quad P\{|\eta_n^{(n)} - t_1| < \epsilon\} \rightarrow 1.$$

In fact, Theorem 2.4 can be replaced by a stronger result that is analogous to the strong law of large numbers, viz. $P\{\lim \eta_n^{(n)} = t_1\} = 1$. The proof of this is similar to that of Theorem 2.4 with the strong law of large numbers applied to the sum $S_n(t)$.

Theorem 2.5:

If $\mu(t)$ has a positive first derivative $\mu'(t)$ at t_1 then $\sqrt{n}(\eta_n^{(n)} - t_1)$ has a limiting normal d.f. with mean zero and variance $\{\sigma(t_1)/\mu'(t_1)\}^2$.

The proof of this theorem depends on:

Lemma 2.7:

If $\mu(t)$ is continuous at some point t_1 , then $\sigma(t)$ is continuous at t_1 .

Proof:

Assume that $\mu(t)$ is continuous at t_1 ; then from the following representation (e.g., see [1], p. 54)

$$(2.16) \quad \sigma^2(t) = 2 \int_0^t \mu(t-x) d\mu(x) + \mu(t) - \{\mu(t)\}^2$$

it suffices to show that the first term in this expression -- denoted by $v(t)$ -- is continuous at t_1 . For small h

$$(2.17) \quad v(t_1 + h) - v(t_1) = \int_{t_1}^{t_1+h} \mu(t_1+h-x) d\mu(x) + \int_0^{t_1} [\mu(t_1+h-x) - \mu(t_1-x)] d\mu(x).$$

The first term in (2.16) is not greater than $\mu(h)[\mu(t_1 + h) - \mu(t_1)]$, which becomes arbitrarily small as $h \rightarrow 0$ by continuity of μ at the point t_1 . The second term will give a nonnegligible contribution only if μ has discontinuities at both $t_1 - x$ and x , for some x , $0 < x < t_1$. However, since

$$(2.18) \quad \mu(t) = \sum_{i=1}^{\infty} F^{(i)}(t),$$

this means that $F^{(i)}(t)$ is discontinuous at $t_1 - x$, for some i , and $F^{(j)}(t)$ is discontinuous at x , for some j , so that $F^{(1+j)}(t)$ is discontinuous at t_1 and this with (2.18) contradicts the hypothesis. Thus, σ is continuous at the point t_1 .

Proof of Theorem 2.5:

For fixed x

$$(2.19) \quad \left\{ \sqrt{n}(\eta_n^{(n)} - t_1) > x \right\} \Leftrightarrow \left\{ S_n(t_1 + x/\sqrt{n}) < n \right\}$$

$$\Leftrightarrow \left\{ \frac{S_n(t_1 + x/\sqrt{n}) - n\mu(t_1 + x/\sqrt{n})}{\sqrt{n}\sigma(t_1 + x/\sqrt{n})} < \frac{n - n\mu(t_1 + x/\sqrt{n})}{\sqrt{n}\sigma(t_1 + x/\sqrt{n})} \right\}$$

Now $S_n(t_1 + x/\sqrt{n})$ may be written in the form $\sum_{k=1}^n X_{nk}$, where

$X_{nk} = N_k(t_1 + x/\sqrt{n})$; it is clear that the $\{X_{nk}\}$ are independent, identically distributed and have finite moments of all orders. Thus, a modification of Liapunov's version of the central limit theorem (see [12], p. 277) may be applied to give:

$$(2.20) \quad P \left\{ \frac{S_n(t_1 + x/\sqrt{n}) - n\mu(t_1 + x/\sqrt{n})}{\sqrt{n}\sigma(t_1 + x/\sqrt{n})} \leq u \right\} \rightarrow \Phi(u)$$

where, as before, Φ is the normal (0,1) d.f.

Now $\mu(t_1 + x/\sqrt{n})$ may be written in the form

$$(2.21) \quad \mu(t_1 + x/\sqrt{n}) = \mu(t_1) + (x/\sqrt{n})\mu'(t_1) + o(1/\sqrt{n}),$$

as $n \rightarrow \infty$. Also, from Lemma 2.7,

$$(2.22) \quad \sigma(t_1 + x/\sqrt{n}) \rightarrow \sigma(t_1).$$

Combining (2.21) and (2.22)

$$(2.23) \quad \frac{n - n\mu(t_1 + x/\sqrt{n})}{\sqrt{n}\sigma(t_1 + x/\sqrt{n})} \rightarrow - \left(\frac{x\mu'(t_1)}{\sigma(t_1)} \right).$$

Since the normal d.f. is continuous, the conclusion of the theorem follows by substituting (2.20) and (2.23) into (2.19).

It should be noted that the proof of the theorem is not sensitive to the assumption of common lifetime d.f. for each of the original and spare components. All that is needed is a central limit theorem to hold for the sum $S_n(t_1 + x/\sqrt{n})$ and convergence of the appropriate sequence of constants as in (2.23).

Examples:

1. Suppose that $F(t) = 1 - e^{-\lambda t}$, so that $\mu(t) = \sigma^2(t) = \lambda t$. Then $\sqrt{n}(\eta_n^{(n)} - \lambda^{-1})$ has a limiting normal d.f. with mean zero and variance λ^{-2} . In fact, this result can be obtained quite simply by observing that the times between consecutive failures are independent, identically distributed exponential random variables.
2. Nonidentical components. Suppose that the original components have lifetime d.f. $F_e(t) = \lambda \int_0^t \bar{F}(x) dx$ and the spares have lifetime d.f. $F(t)$, where $F(t) = 1 - (1 + 2\lambda t)e^{-2\lambda t}$ is a gamma d.f. Thus, the sequence of failures in each socket corresponds to an equilibrium renewal process, so that $\mu(t) = \lambda t$ and $\sigma^2(t) = \lambda t/2 + 1/8 e^{-4\lambda t}$.

Then $\sqrt{n}(\eta_n^{(n)} - \lambda^{-1})$ has a limiting normal d.f. with mean zero and variance $(1/2 + 1/8 e^{-4})\lambda^{-2}$.

k-out-of-n Structures

The methods of this section can be applied to more general types of systems with replaceable components. For example, consider a k-out-of-n system with m spares where, as before, the component lifetimes are assumed to be independent and identically distributed. As components fail, they are immediately detected and replaced by new components until m replacements have been made; the system fails when k additional failures have occurred, i.e., k + m in all. Let the system lifetime be denoted by $\zeta_{nk}^{(m)}$.

Lemma 2.8:

If $k + m \leq n$, then

$$\xi_n^{(k+m)} \underset{\text{st}}{\geq} \zeta_{nk}^{(m)} \underset{\text{st}}{\geq} \xi_n^{(k)}$$

Proof:

The first part of the inequality is proved as in Lemma 2.3, and the second part by observing that a system with spares survives longer than a system without spares.

Lemma 2.9:

If $k + m \leq n$, then

$$\xi_n^{(k+m)} \underset{\text{st}}{\geq} \zeta_{nk}^{(m)} \underset{\text{st}}{\geq} \eta_n^{(k+m)}$$

Proof:

Only the second part remains to be proved; this follows by noting that between the m^{th} and $(k + m)^{\text{th}}$ failures there are fewer than n components liable to failure and so system failure is stochastically larger than in the case where replacements are continually available.

Making certain assumptions about the behaviour of $k = k(n)$ and $m = m(n)$ as $n \rightarrow \infty$ enables some deductions to be made concerning the limiting d.f.'s of $a_n^{-1}(\zeta_{nk}^{(m)} - b_n)$. For example, consider the two cases:

- (i) $m/n \rightarrow 0$, $k/n \rightarrow \lambda$, $0 < \lambda < 1$. By using Lemma 2.8 and the results of Smirnov [14] concerning limit d.f.'s of central order statistics, the limiting d.f.'s of system lifetime may be completely characterised.
- (ii) $(k + m)$ finite or $(k + m) \sim cn^\alpha$, $c > 0$, $0 < \alpha < 1/2$. Then Lemma 2.9 and Theorems 2.1 and 2.2 enable one to describe completely the possible limiting d.f.'s.

3. ORDER STATISTICS FROM A NONHOMOGENEOUS SAMPLE

In this section, it is shown that the limiting d.f.'s for order statistics of fixed rank from an independent but not identically distributed sample are essentially derived from the limiting d.f. of the minimum as in the identical case. Suppose that X_k has d.f. $F_k(x)$, then Juncosa [10] has shown

Lemma 3.1:

If $\Lambda(x)$ is a positive, nonconstant, nondecreasing function such that for suitable $\{a_n > 0\}$ and $\{b_n\}$:

$$(3.1) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n F_k(a_n x + b_n) = \Lambda(x)$$

and

$$\max_{1 \leq k \leq n} F_k(a_n x + b_n) \rightarrow 0, \quad \forall x,$$

then

$$\lim_{n \rightarrow \infty} P\{\xi_n > a_n x + b_n\} = \lim_{n \rightarrow \infty} \prod_{k=1}^n \bar{F}_k(a_n x + b_n) = \exp[-\Lambda(x)].$$

Theorem 3.1:

If the conditions of Lemma 3.1 are satisfied, and m is fixed, then

$$\lim_{n \rightarrow \infty} P\left\{\xi_n^{(m)} > a_n x + b_n\right\} = \left\{ \sum_{j=0}^{m-1} \frac{\Lambda^j(x)}{j!} \right\} \exp[-\Lambda(x)].$$

The following lemmas are needed:

Lemma 3.2:

If $u_{kn} \geq 0$, $v_{kn} \geq 0$ are such that:

(i) $u_{kn} \rightarrow u$, uniformly in k , $1 \leq k \leq n$, and

(ii) $\sum_{k=1}^n v_{kn} \rightarrow v$, as $n \rightarrow \infty$, where u and v are finite, then:

$$(3.2) \quad \sum_{k=1}^n u_{kn} v_{kn} \rightarrow uv, \text{ as } n \rightarrow \infty.$$

Lemma 3.3:

If $d_{kn} \geq 0$ is such that $\max_{1 \leq k \leq n} d_{kn} \rightarrow 0$ and $\sum_{k=1}^n d_{kn} \rightarrow \Lambda(x)$, as $n \rightarrow \infty$, then:

$$(3.3) \quad \lim \sum' d_{k_1 n} d_{k_2 n} \dots d_{k_j n} = \frac{\Lambda^j(x)}{j!}$$

where \sum' is a sum over all k_1 such that $1 \leq k_1 < k_2 < \dots < k_j \leq n$.

Proof:

The result holds for $j = 1$, so assume the inductive hypothesis that it holds up to $j - 1$. Clearly

$$(3.4) \quad \left\{ \sum_{k=1}^n d_{kn} \right\}^j \rightarrow \Lambda^j(x).$$

Also

$$\left\{ \sum_{k=1}^n d_{kn} \right\}^j = j! \sum' d_{k_1 n} d_{k_2 n} \dots d_{k_j n} + \left\{ \begin{array}{l} \text{Terms containing} \\ \text{squares and higher} \\ \text{powers of } d_{kn} \end{array} \right\}$$

It is easy to see that the second term above is asymptotically negligible.

For example:

$$\sum' d_{k_1 n}^2 d_{k_2 n} \dots d_{k_{j-1} n} \leq \max_{1 \leq k \leq n} d_{kn} \cdot \sum' d_{k_1 n} d_{k_2 n} \dots d_{k_{j-1} n}$$

and the right hand side tends to zero from the inductive hypothesis and the assumptions of the lemma. The conclusion follows by comparison of 3.4 and 3.5.

Proof of Theorem 3.1:

Let:

$$(3.6) \quad \bar{G}_{nm}(x) = P\left\{\xi_n^{(m)} > x\right\} = \sum_{j=0}^{m-1} A_{jn}(x),$$

where

$$(3.7) \quad A_{jn}(x) = \sum_{k=1}^* \prod_{k=1}^n F_k^{i_k}(x) \bar{F}_k^{1-i_k}(x),$$

and \sum^* is the summation over all terms such that $i_1 + i_2 + \dots + i_n = j$ and $i_k = 0, 1$. Now $A_{jn}(a_n x + b_n)$ may be written in the form $\sum^* u_{rn} v_{rn}$ where

$$(3.8) \quad u_{rn} = \prod_{k=1}^n \bar{F}_k^{1-i_k}, \quad v_{rn} = \prod_{k=1}^n F_k^{i_k},$$

r is a symbol for the partition $r = (i_1, i_2, \dots, i_n)$, and the argument is omitted but understood to be $a_n x + b_n$. Since u_{rn} differs from $\prod_{k=1}^n \bar{F}_k(a_n x + b_n)$ by only a finite number of factors, Lemma 3.1 implies

$$(3.9) \quad u_{rn}(a_n x + b_n) \rightarrow \exp. [-\Lambda(x)], \quad \text{uniformly in } r.$$

If d_{kn} is identified with $F_k(a_n x + b_n)$, the conditions of Lemma 3.3 are satisfied, and so

$$(3.10) \quad \sum^* v_{rn}(a_n x + b_n) \rightarrow \frac{\Lambda^j(x)}{j!}.$$

Application of Lemma 3.2 together with 3.9 and 3.10 then gives:

$$(3.11) \quad A_{jn}(a_n x + b_n) \rightarrow \exp[-\lambda(x)] \cdot \frac{\lambda^j(x)}{j!}.$$

The proof of the theorem is complete.

4. ORDER STATISTICS FROM A SAMPLE OF RANDOM SIZE

Consider a sample $(X_1, X_2, \dots, X_{N_n})$ of independent random variables with common d.f. $F(x)$, where N_n is a random variable distributed independently of the $\{X_i\}$. Assuming that $N_n \rightarrow \infty$ in probability as $n \rightarrow \infty$, the order statistics $\xi_n^{(m)}$ are defined with probability one for fixed m as $n \rightarrow \infty$. To avoid writing conditional expectations it is assumed below, without loss of generality, that $N_n \geq m$. Theorems 4.1, 4.2 and 4.3 characterize the limiting d.f.'s of these order statistics and extend the results of Berman [2] who obtained the limiting d.f.'s of the maximum under such conditions. Let:

$$(4.1) \quad P\{n^{-1}N_n \leq y\} = A_n(y)$$

and assume that there exists a d.f. $A(y)$, which may be improper, such that for all y in its continuity set:

$$(4.2) \quad \lim_{n \rightarrow \infty} A_n(y) = A(y).$$

To ensure that the limiting d.f.'s are proper, $A(y)$ will be required to satisfy one of:

$$(4.3) \quad A(0+) - A(0-) = 0 ; A(\infty) - A(0-) > 0$$

$$(4.4) \quad A(0+) - A(0-) < 1 ; A(\infty) - A(0-) = 1$$

$$(4.5) \quad A(0+) - A(0-) = 0 ; A(\infty) - A(0-) = 1.$$

Theorem 4.1:

If $F(x)$ is in the domain of attraction of $\phi_{(1)}(x)$, then $a_n^{-1}(\xi_n^{(m)} - b_n)$ has a limiting d.f. $G(x)$ which is necessarily of the type:

$$G(x) = 0$$

for $x \leq 0$

$$= 1 - \sum_{j=0}^{m-1} \int_0^{\infty} \frac{(yx^\alpha)^j}{j!} \exp[-yx^\alpha] dA(y) \quad \text{for } x > 0, \alpha > 0$$

where $A(y)$ satisfies condition (4.3).

Proof:

Let $G_{nm}(a_n x + b_n) = P\left\{a_n^{-1}(\xi_n^{(m)} - b_n) \leq x\right\}$, so that

$$(4.6) \quad \bar{G}_{nm}(a_n x + b_n) = E_N \left\{ \sum_{j=0}^{m-1} \binom{N}{j} \bar{F}^{N-j}(a_n x + b_n) F^j(a_n x + b_n) \right\}$$

where $E_N(\cdot)$ denotes expectation with respect to N and the constants $\{a_n\}$ and $\{b_n\}$ are such that:

$$(4.7) \quad \lim \bar{F}^n(a_n x + b_n) = \phi_{(1)}(x), \quad \forall x.$$

It is easy to show that $\lim G_{nm}(a_n x + b_n) = 0$ for $x \leq 0$; consider one of the terms in the sum (4.6) with $x > 0$:

$$(4.8) \quad E_N \left\{ \binom{N}{j} \bar{F}^{N-j}(a_n x + b_n) F^j(a_n x + b_n) \right\} = \\ = \frac{1}{j!} \left\{ \frac{n F(a_n x + b_n)}{\bar{F}(a_n x + b_n)} \right\}^j E_N \left\{ \frac{N(N-1)\dots(N-j+1)}{n^j} \bar{F}^N(a_n x + b_n) \right\}.$$

From (4.7), $\lim \bar{F}(a_n x + b_n) = 1$ and $\lim n F(a_n x + b_n) = -\log \phi_{(1)}(x)$, so for the first factor in (4.8)

$$(4.9) \quad \left\{ \frac{n F(a_n x + b_n)}{\bar{F}(a_n x + b_n)} \right\}^j \rightarrow \{-\log \phi_{(1)}(x)\}^j.$$

Now

$$\begin{aligned}
 \lim E_N \left\{ n^{-j} N(N-1) \dots (N-j+1) e^{-sn^{-1}N} \right\} &= \lim E_N \left\{ (n^{-1}N)^j e^{-sn^{-1}N} \right\} \\
 (4.10) \qquad \qquad \qquad &= \lim \int_0^\infty y^j e^{-sy} dA_n(y) \\
 &= \int_0^\infty y^j e^{-sy} dA(y)
 \end{aligned}$$

for $s > 0$, by the extended Helly-Bray lemma. What has been shown so far is:

$$(4.11) \qquad \int_0^\infty g(y) dA_n(y) \rightarrow \int_0^\infty g(y) dA(y)$$

where $g(y) = y^j \exp[-sy]$; now let $g_n(y) = y^j \exp[ny \log \bar{F}(a_n x + b_n)]$. Then since $-n \log \bar{F}(a_n x + b_n) \rightarrow -\log \phi_{(1)}(x) > 0$, for $x > 0$, it may be shown that $g_n(y) \rightarrow g(y)$ uniformly in y for each fixed $x > 0$ as $n \rightarrow \infty$, with $s = -\log \phi_{(1)}(x)$. Hence

$$(4.12) \qquad \int_0^\infty g_n(y) dA_n(y) \rightarrow \int_0^\infty g(y) dA(y)$$

which is the limit of the second factor in (4.8); combining this with (4.6), (4.9), (4.12) and (1.1), the main part of the theorem follows.

Uniqueness of the limit d.f. follows from Khintchine's theorem [7] that a sequence of d.f.'s $\{F_n(a_n x + b_n)\}$ can have at most one proper limit type. (4.3) follows from the conditions $G(0+) < 1$ and $G(\infty) = 1$ that the limit d.f. be proper.

Theorem 4.2:

If $F(x)$ is in the domain of attraction of $\phi_{(2)}(x)$, then $a_n^{-1}(\xi_n^{(m)} - b_n)$ has a limiting d.f. $G(x)$ which is necessarily of the type

$$G(x) = 1 - \sum_{j=0}^{m-1} \int_0^{\infty} \frac{\{y(-x)^{-\alpha}\}^j}{j!} \exp[-y(-x)^{-\alpha}] dA(y) \quad \text{for } x < 0$$

$$= 1 \quad \text{for } x \geq 0, \alpha > 0$$

where $A(y)$ satisfies condition (4.4).

Theorem 4.3:

If $F(x)$ is in the domain of attraction of $\phi_{(3)}(x)$, then $a_n^{-1}(\xi_n^{(m)} - b_n)$ has a limiting d.f. $G(x)$ which is necessarily of the type

$$G(x) = 1 - \sum_{j=0}^{m-1} \int_0^{\infty} \frac{(ye^x)^j}{j!} \exp[-ye^x] dA(y) \quad -\infty < x < \infty$$

where $A(y)$ satisfies condition (4.5).

The proofs of Theorems 4.2 and 4.3 are analogous to that of Theorem 4.1. Conditions (4.4) and (4.5) follow by considering $G(0-)$, $G(-\infty)$ and $G(-\infty)$, $G(+\infty)$ respectively.

5. APPLICATION TO SUPERPOSITION OF RENEWAL PROCESSES

Necessary and sufficient conditions for a sum of a large number of renewal processes to be close to a (nonhomogeneous) Poisson process have been given by Grigelionis [8]. He considers n general renewal processes, where $\hat{F}_{nr}(x)$ and $F_{nr}(x)$ are respectively the d.f.'s of time to the first event and between subsequent events, $1 \leq r \leq n$. Under the condition $\lim_{n \rightarrow \infty} \max_r \max \{\hat{F}_{nr}(x), F_{nr}(x)\} = 0$, for all x , he proves

Theorem 5.1:

For the superposed process to converge as $n \rightarrow \infty$ to a Poisson process with parameter $\Lambda(x)$, it is necessary and sufficient that

$$\sum_{r=1}^n \hat{F}_{nr}(x) \rightarrow \Lambda(x) \quad \forall x > 0.$$

It is of interest to know what form the function $\Lambda(x)$ may take when the underlying renewal processes are identical and $\hat{F}_{nr}(x) = F_{nr}(x) = F(a_n x + b_n)$, $1 \leq r \leq n$, where $\{a_n > 0\}$ and $\{b_n\}$ are suitably chosen constants. $F(x)$ is a fixed d.f. on the nonnegative real axis, and it is convenient to assume that $F(\epsilon) > 0$ for $\epsilon > 0$ and take $b_n = 0$. The following result is suggested by Theorem 2.1.

Theorem 5.2:

If the sum of n identical renewal processes converges to a Poisson process with parameter $\Lambda(x)$, then $\Lambda(x) = ax^\alpha$ for some $a > 0$, $\alpha > 0$. Moreover, $F(x)$ is in the domain of attraction of $\phi_{(1)}(x)$.

Proof:

Observe that the first event in the sum process is the same as the minimum of a set of n independent and identically distributed random variables, so that Gnedenko's results show that this minimum has a limiting d.f. which is one of those in (1.1). However, the d.f. $\phi_{(2)}(x)$ can be ruled out because its domain of attraction contains only d.f.'s corresponding to random variables unbounded below [6]. Moreover, $\phi_{(3)}(x)$ is a d.f. on the whole real axis and so cannot generate a renewal process. Thus the limiting d.f. of the minimum must have a d.f. of the same type as $\phi_{(1)}(x)$ which corresponds to a Poisson process with parameter $\Lambda(x) = ax^\alpha$.

In fact, the same form for $\Lambda(x)$ holds under wider conditions than those above. Suppose $F(x)$ is in the domain of attraction of $\phi_{(1)}(x)$ —see Section 1— so that

$$(5.1) \quad \lim_{x \rightarrow 0+} F(tx) / F(x) = t^\alpha, \quad \alpha > 0.$$

Let

$$(5.2) \quad \hat{F}_{nr}(x) = F_{nr}(x) = F(\gamma_r a_n x),$$

so that in the terminology of Section 2 one may think of identical components which are subjected to different levels of stress in different sockets.

Assuming that

$$(5.3) \quad \sum_{r=1}^{\infty} \gamma_r^\alpha = \infty \quad \text{and} \quad \gamma_n = o\left(\sum_{r=1}^n \gamma_r^\alpha\right),$$

then if a_n are chosen to satisfy

$$(5.4) \quad a_n = \left\{ \inf x \mid F(x) \geq 1 / \left(\sum_{r=1}^n \gamma_r^\alpha \right) \right\},$$

it may be shown (see [10]) that

$$(5.5) \quad \sum_{r=1}^n F(\gamma_r a_n x) \rightarrow x^\alpha.$$

Hence, Grigelionis' conditions are satisfied, and one has:

Theorem 5.3:

The sum of n renewal process of the form specified by (5.1), (5.2) and (5.3) converges to a Poisson process with parameter $\Lambda(x) = ax^\alpha$.

Now suppose that the number of renewal processes is itself a random variable N_n analogous to the case of order statistics in Section 4. Using the notation of that section:

Theorem 5.4:

The sum of a random number N_n of identical renewal process of the form $\hat{F}_{nr}(x) = F_{nr}(x) = F(a_n x)$, where $F(x)$ satisfies (5.1) and $\{a_n > 0\}$ are suitable constants, converges to a counting process as $n \rightarrow \infty$ in which the d.f. $G_m(x)$ of the time to the m^{th} event is given by

$$\begin{aligned} G_m(x) &= 0 && \text{for } x \leq 0 \\ &= 1 - \sum_{j=0}^{m-1} \int_0^\infty \frac{(yx^\alpha)^j}{j!} \exp[-yx^\alpha] dA(y) && \text{for } x > 0 \end{aligned}$$

where $A(y) = \lim P\{n^{-1}N_n \leq y\}$ satisfies

$$(5.5) \quad A(0+) - A(0-) = 0; \quad A(\infty) - A(0-) = 1.$$

Proof:

A corresponding result for order statistics from a random sample has been proved in Theorem 4.1; it remains to be shown that the d.f. of the m^{th} order statistic in such a random sample is asymptotically the same as the d.f. of the m^{th} event in the superposed process. From (5.5), k and K may be chosen, $0 < k < K < \infty$, so that $P\{k \leq n^{-1}N_n \leq K\} > 1 - \epsilon$, where $\epsilon > 0$ is arbitrary. Then for each N_n in this interval the asymptotic closeness of the two limiting d.f.'s in question may be shown as in the proof of Theorem 2.3. Unconditioning on N_n and letting $\epsilon \rightarrow 0$, the desired result is obtained.

Theorem 5.4 shows that randomness of the number of renewal processes leads to counting processes which are mixtures of (nonhomogeneous) Poisson processes.

Example:

Take $\alpha = 2$, $A(y) = 1 - e^{-y}$. The limiting d.f.'s are

$$\begin{aligned} G_m(x) &= 0 && \text{for } x \leq 0 \\ &= \{x^2/(1+x^2)\}^m && \text{for } x > 0 \end{aligned}$$

6. COMPOSITION OF COHERENT STRUCTURES

Many multi-component systems occurring in the field of reliability are coherent structures (e.g., see [1]). A coherent structure consisting of n components c_1, c_2, \dots, c_n is described by its structure function $\phi(\underline{x})$ which is 1 if the system works and 0 otherwise, where $\underline{x} = (x_1, x_2, \dots, x_n)$ and x_i is 1 if component c_i is working and 0 if it is failed. The binary function ϕ is required to satisfy

- (i) Each component c_i must be *essential*; that is to say, there is a realization $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ such that

$$\phi(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = 1 \text{ and}$$

$$\phi(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0.$$
- (ii) $\phi(\underline{x}) \leq \phi(\underline{y})$ for all $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$ such that $x_i \leq y_i, 1 \leq i \leq n$.

From (i) and (ii) it follows immediately that $\phi(\underline{1}) = 1$ and $\phi(\underline{0}) = 0$, where $\underline{1} = (1, 1, \dots, 1)$ and $\underline{0} = (0, 0, \dots, 0)$.

Most of the classical problems in extreme value theory are contained in the following general question: "Given a coherent structure with a finite number of components and some procedure to increase the number of components without bound, what are the possible limiting d.f.'s for the system lifetime for given component lifetime d.f.'s?" For example, the minimum of a set of random variables corresponds to the lifetime of a series structure, and the maximum likewise corresponds to a parallel structure. The so-called k -out-of- n structure generate order statistics and series-parallel and parallel-series systems give rise to the minimax and max/min (see [3]) of double arrays of random variables.

In this section, results are obtained when the method of "expanding" the structure is that of repeated composition. Thus, starting with a system of

n components, each component is replaced by a similar system giving a system of n^2 components, then each component in this system is replaced by a replica of the original system, giving a system of n^3 components, etc. The first three stages of composition for a three component system are illustrated in Figure 6.1.

It is assumed that all components are independent and identical. In particular, let $p = P\{X_1 = 1\}$ be the *reliability* of each component c_1 , where X_1 is the binary random variable designating the state of component c_1 . The structure function $\phi(x)$ becomes a binary random variable and the reliability of the structure will be denoted by $h(p) = P\{\phi(X) = 1\}$. Now suppose that the components fail over time and have a common lifetime d.f. $F(t)$ with binary indicator random variables $X_1(t)$ which are 1 or 0 according as component c_1 is in a working or failed state respectively at time t . Then $P\{X_1(t) = 1\} = \bar{F}(t)$ and so the system has lifetime d.f. $1 - h(\bar{F}(t))$. The system which results after n compositions has lifetime d.f. $1 - h^{(n)}(\bar{F}(t))$, where $h^{(n)}$ denotes the n -fold composition of the function h with itself. It is assumed throughout this section that $h(p) \neq p$. Suppose that for suitable $\{a_n > 0\}$ and $\{b_n\}$

$$(6.1) \quad h^{(n)}(\bar{F}(a_n x + b_n)) \rightarrow \bar{G}(x) .$$

Then

Lemma 6.1:

If $G(x)$ is a limiting d.f., then $h(\bar{G}(x)) = \bar{G}(\alpha x + \beta)$ for some $\alpha > 0$, β .

Remark:

This lemma generalizes the concept of *maximum stable* d.f.'s introduced by Gnedenko. The relation $h^{(n)}(\bar{G}(x)) = \bar{G}(\alpha_n x + \beta_n)$ for some $\alpha_n > 0$, β_n

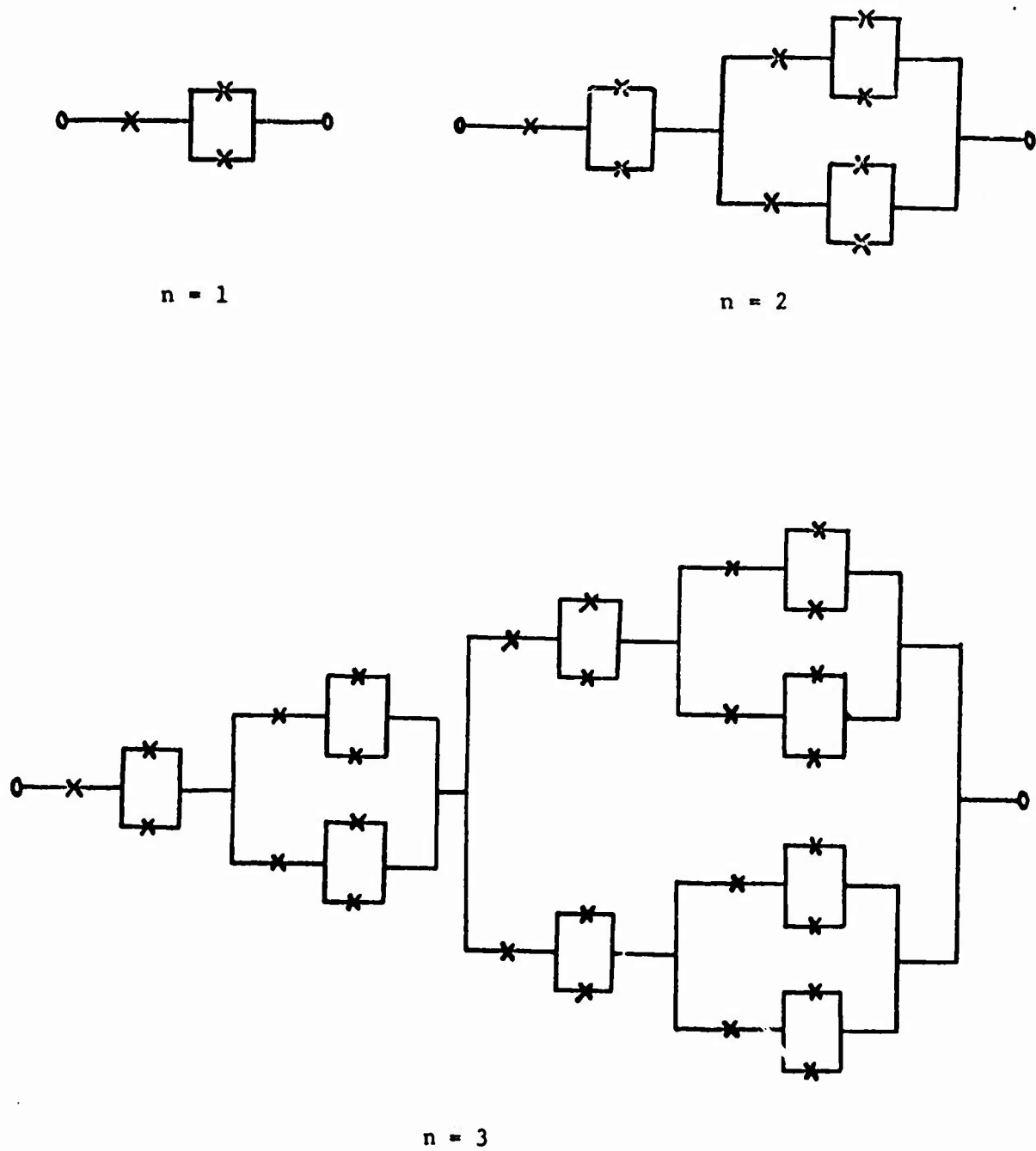


Figure 6.1

and for all n is no more general, since it is implied by the lemma with $\alpha_n = \alpha^n$ and $\beta_n = (1 + \alpha + \dots + \alpha^{n-1})\beta$.

Proof:

From (6.1),

$$h^{(n+1)}(\bar{F}(a_n x + b_n)) \rightarrow h(\bar{G}(x))$$

and

$$h^{(n+1)}(\bar{F}(a_{n+1} x + b_{n+1})) \rightarrow \bar{G}(x).$$

But by Khintchine's theorem [7], these two d.f.'s must be of the same type, i.e.,

$$h(\bar{G}(x)) = \bar{G}(\alpha x + \beta).$$

If $\alpha \neq 1$, set $G^*(x) = G\left(x + \frac{\beta}{1 - \alpha}\right)$, so that

$$(6.2) \quad h(\bar{G}^*(x)) = h\left(\bar{G}\left(x + \frac{\beta}{1 - \alpha}\right)\right) = \bar{G}\left(\alpha x + \frac{\beta}{1 - \alpha}\right) = \bar{G}^*(\alpha x).$$

If $\alpha = 1$, set $G^*(x) = G(\log x)$ and $\alpha^* = e^\beta$, so that

$$(6.3) \quad h(\bar{G}^*(x)) = h(\bar{G}(\log x)) = \bar{G}(\log x + \log \alpha^*) = \bar{G}^*(\alpha^* x).$$

Theorem 6.1:

$G(x)$ is a limiting d.f. if and only if $\bar{G}(x)$ is type equivalent to $\psi(x)$ or $\psi(e^x)$, where $1 - \psi(x)$ is a d.f. and

$$(6.4) \quad h(\psi(x)) = \psi(\alpha x) \quad \text{for all } x, \text{ where } \alpha > 0, \alpha \neq 1.$$

Proof:

The necessity part follows from (6.2), (6.3) and Lemma 6.1. For sufficiency, notice that each such d.f. is in its own domain of attraction by choosing.

$$a_n = \alpha^{-n}, b_n = 0 \text{ or } a_n = 1, b_n = -n \log \alpha \text{ respectively.}$$

The following is a well-known result (e.g., see [1], p. 199).

Theorem 6.2:

Excluding the trivial case $h(p) = p$, the reliability function $h(p)$ is one of three kinds

- (i) $h(p) < p$ for $0 < p < 1$
- (ii) $h(p) > p$ for $0 < p < 1$
- (iii) $h(p) < p$ for $0 < p < p_0$
 $= p_0$ for $p = p_0$ (h is "S-shaped about p_0 ")
 $> p$ for $p_0 < p < 1$

Defining $h^*(p) = 1 - h(1 - p)$ and $\psi^*(x) = 1 - \psi(-x)$, it is seen that $h(p) < p \Leftrightarrow h^*(p) > p$ and

$$(6.5) \quad h^*(\psi^*(x)) = 1 - h(\psi(-x)) = 1 - \psi(-\alpha x) = \psi^*(\alpha x),$$

so that it suffices to consider cases (i) and (iii) only.

Lemma 6.2:

If $h(p) < p$ and $\psi(x)$ satisfies (6.4), then neither of the following situations can occur

- (i) $\alpha > 1, \bar{x} \leq 0, 0 < \psi(\bar{x}) < 1.$
- (ii) $\alpha < 1, \bar{x} \geq 0, 0 < \psi(\bar{x}) < 1.$

Proof:

- (i) $\psi(\alpha\bar{x}) \geq \psi(\bar{x})$, since $\alpha\bar{x} \leq \bar{x}$ and ψ is monotone, and $h(\psi(\bar{x})) < \psi(\bar{x})$ by the assumptions. Thus with (6.4), a contradiction results. The proof of (ii) is similar. Likewise, one can show

Lemma 6.3:

If $h(p)$ is S-shaped about p_0 and $\psi(x)$ satisfies (6.4), then none of the following can hold

- (i) $\alpha < 1$, $\bar{x} \geq 0$, $0 < \psi(\bar{x}) < p_0$.
(ii) $\alpha < 1$, $\bar{x} \leq 0$, $p_0 < \psi(\bar{x}) < 1$.
(iii) $\alpha > 1$, $\bar{x} \leq 0$, $0 < \psi(\bar{x}) < p_0$.
(iv) $\alpha > 1$, $\bar{x} \geq 0$, $p_0 < \psi(\bar{x}) < 1$.

From (i), (ii) and the fact that $\psi(x)$ is the tail of a proper d.f., it is seen that (6.4) cannot hold with $\alpha < 1$ when $h(p)$ is S-shaped. Thus the solutions to (6.4) take one of the following five forms

- 1a) $h(p) < p$, $\alpha > 1$, $\psi(x) = 1$ for $x \leq 0$
 $0 < \psi(x) < 1$ for $x > 0$.
1b) $h(p) < p$, $\alpha < 1$, $0 < \psi(x) < 1$ for $x < 0$
 $\psi(x) = 0$ for $x \geq 0$.
2a) $h(p)$ S-shaped about p_0 , $\alpha > 1$, $\psi(x) = 1$ for $x \leq 0$
 $0 < \psi(x) < p_0$ for $x > 0$.
2b) $h(p)$ S-shaped about p_0 , $\alpha > 1$, $p_0 < \psi(x) < 1$ for $x < 0$
 $\psi(x) = 0$ for $x \geq 0$.
2c) $h(p)$ S-shaped about p_0 , $\alpha > 1$, $p_0 < \psi(x) < 1$ for $x < 0$
 $0 < \psi(x) < p_0$ for $x > 0$.

The class of all solutions to (6.4) can now be generated; for example:

Theorem 6.3:

The class of all solutions of the form 1a) may be obtained by the following procedure:

- (i) Pick any $\alpha > 1$.
- (ii) Pick any $x_0 > 0$.
- (iii) Define $\psi(x)$ in the half-open, half-closed interval $(x_0, \alpha x_0]$ so that it is nonincreasing and not constant over the interval.
- (iv) Extend $\psi(x)$ to the positive real axis by defining

$$\psi(x) = h^{(k)} \left\{ \psi \left(\alpha^{-k} x \right) \right\}, \text{ for } \alpha^k x_0 < x \leq \alpha^{k+1} x_0, \text{ where a negative index } k \text{ means the } (-k)^{\text{th}} \text{ composition of the inverse function } h^{-1}.$$

Similar procedures may be derived for solutions which are of the other forms.

It is clear that the class of limit d.f.'s thus generated is very large. In the case of composition of series structures, the class of limit d.f.'s has been characterized by Mejlzer [13] and includes d.f.'s other than those of the type (1.1). For example, the function

$$(6.6) \quad \begin{aligned} \psi(x) &= 0 && \text{for } x \leq 0 \\ &= \exp[-x^3(5 + \sin(2\pi \log x / \log 2))] && \text{for } x > 0 \end{aligned}$$

is the tail of a d.f. and satisfies (6.4) for $h(p) = p^8$, $\alpha = 2$.

REFERENCES

- [1] Barlow, R. E. and F. Proschan, MATHEMATICAL THEORY OF RELIABILITY, John Wiley and Sons, New York, (1965).
- [2] Berman, S. M., "Limiting Distribution of the Maximum Term in Sequences of Dependent Random Variables," Ann.Math.Statist., Vol. 33, pp. 894-908, (1962).
- [3] Chernoff, H. and H. Teicher, "Limit Distributions of the Minimax of Independent Identically Distributed Random Variables," Trans.Amer.Math.Soc., Vol. 116, pp. 474-491, (1965).
- [4] Chibisov, D. M., "On Limit Distributions for Order Statistics," Theor.Prob.Appl., Vol. 9, pp. 142-148, (1964).
- [5] Epstein, B., "Elements of the Theory of Extreme Values," Technometrics, Vol. 2, pp. 27-41, (1960).
- [6] Gnedenko, B. V., "Sur la Distribution Limite du Terme Maximum d'une Série Aléatoire," Ann.Math., Vol. 44, pp. 423-453, (1943).
- [7] Gnedenko, B. V. and A. Kolmogorov, LIMIT DISTRIBUTIONS FOR SUMS OF INDEPENDENT RANDOM VARIABLES, Addison-Wesley, Cambridge, Massachusetts, (1954).
- [8] Grigelionis, B. I., "Limit Theorems for Sums of Renewal Processes," CYBERNETICS IN THE SERVICE OF COMMUNISM, Vol. 2, RELIABILITY THEORY AND QUEUEING THEORY, Collected articles (in Russian), "Energy" Publishing House, Moscow-Leningrad, pp. 246-266, (1964).
- [9] Gumbel, E. J., STATISTICS OF EXTREMES, Columbia University Press, New York, (1958).
- [10] Juncosa, M. L., "On the Distribution of the Minimum in a Sequence of Mutually Independent Random Variables," Duke Math.J., Vol. 16, pp. 609-618, (1949).
- [11] Kolmogorov, A., GRUNDBEGRIFFE DER WAHRSCHEINLICHKEITSRECHNUNG, Berlin, (1933).
- [12] Loève, M., PROBABILITY THEORY, Van Nostrand, New York, (1955).
- [13] Mejsler, D., "On a Certain Class of Limit Distributions and Their Domain of Attraction," Trans.Amer.Math.Soc., Vol. 117, pp. 205-236, (1965).
- [14] Smirnov, N. V., "Limit Distributions for the Terms of a Variational Series," Amer.Math.Soc.Transl., Series 1, No. 67, (1952).
- [15] Watson, G. S., "Extreme Values in Samples from m-Dependent Stationary Stochastic Processes," Ann.Math.Statist., Vol. 25, pp. 798-800 (1954).

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